Inequality

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Let a, b, c be real numbers such that $a^2+b^2+c^2=1$. Prove that

$1/(5-6bc)+1/(5-6ca)+1/(5-6ab) \le 1$.

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Since $\sum \dfrac{1}{5-6bc} \le \sum \dfrac{1}{5-6|b||c|}$ then we can assume for further that $a, b, c \ge 0$.

Let $s := a+b+c, p := ab+bc+ca, q := abc$. Then

$$2p = s^2-1,\ q \le \frac{s^3}{27}, \qquad (3sq \le p^2 \le 1)$$

$9q \ge 4sp - s^3 = 2(s^2-1)s - s^3 = s(s^2-2),\ s^2 = (a+b+c)^2 \le 3(a^2+b^2+c^2) = 3$

and $\sum(5-6ca)(5-6ab) = 75-30(s^2-1)+36sq = 105-30s^2+36qs$,

$(5-6bc)(5-6ca)(5-6ab) = 125+180sq-75(s^2-1)-216q^2 =$

$200-75s^2+180qs-216q^2$ and inequality of the problem becomes

$105-30s^2+36qs \le 200-75s^2+180qs-216q^2 \iff$

(1) $\quad 0 \le h(s,q)$, where $h(s,q) := 95-45s^2+144qs-216q^2$.

We already have upper bound $\dfrac{s^3}{27}$ for q and lower bound for q which we need for further we will obtain using Schure Inequality $\sum a^2(a-b)(a-c) \ge 0$ which in s, q-notation and normalization $a^2+b^2+c^2=1$ becomes $q \ge \dfrac{(s^2+1)(s^2-2)}{12s}$.

Thus, $q \in [q_*, q^*]$ where $q^* = \dfrac{s^3}{27}$ and $q_* := \min\left\{0, \dfrac{(s^2+1)(s^2-2)}{12s}\right\}$ and

$\min\limits_{q\in[q_*,q^*]} h(s,q) = \min\{h(s,q_*), h(s,q^*)\}$ (because $h(s,q)$ as function of q is concave up)

and since our aim to prove inequality (**1**) for any s, q such that $0 < s \le \sqrt{3}$ and $q_* \le q \le q^*$ suffices to prove $h(s,q^*) \ge 0$ and $h(s,q_*) \ge 0$ for $0 < s \le \sqrt{3}$.

We have $h(s,q^*) = 95$

$$-45s^2+144s \cdot \frac{s^3}{27} - 216 \cdot \left(\frac{s^3}{27}\right)^2 = \frac{1}{27}(3-s^2)(855+8s^4-120s^2) \ge 0$$

because $s^2 \le 3$ and $855+8s^4-120s^2 > 0$ for $0 < s \le \sqrt{3}$.

For calculation $h(s,q_*)$ we will consider two cases:

1. If $s \in \left[\sqrt{2}, \sqrt{3}\right]$ then $q_* = \dfrac{(s^2+1)(s^2-2)}{12s}$ and denoting for convenience $t := s^2$

we obtain $h(s,q_*) = 95-45s^2+144s \cdot \dfrac{(s^2+1)(s^2-2)}{12s} - 216\left(\dfrac{(s^2+1)(s^2-2)}{12s}\right)^2 =$

$$95-45t+12(t+1)(t-2) - \frac{3(t+1)^2(t-2)^2}{2t} = \frac{(3-t)(3t^3-21t^2+42t-4)}{2t} \ge 0$$

(because for $t \in [2,3]$ we have $3t^3-21t^2+42t-4 = 3t(-7t+t^2+14)-4 =$

$3t\left((t-7/2)^2+\dfrac{7}{4}\right)-4 > 3 \cdot 2 \cdot \dfrac{7}{4} - 4 = \dfrac{13}{2} > 0$.

2. If $0 < s < \sqrt{2}$ then $h(s,q_*) = h(s,0) = 95-42s^2 > 0$.